RATIONAL CURVES AND PROLONGATIONS OF G-STRUCTURES

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Abstract. In a joint work with N. Mok in 1997, we proved that for an irreducible representation $G \subset \mathbf{GL}(V)$, if a holomorphic G-structure exists on a uniruled projective manifold, then the Lie algebra of G has nonzero prolongation. Using a different approach, we generalize this to an arbitrary connected algebraic subgroup $G \subset \mathbf{GL}(V)$ and a complex manifold containing an immersed rational curve. We also prove a partial converse: a construction of a holomorphic G-structure on a homogeneous complex manifold containing smooth rational curves, under the condition that G has no nonzero fixed vector in V and the prolongation of the Lie algebra of G is finite-dimensional and nonzero.

Keywords. Prolongation of G-structure, Rational curves
AMS Classification. 14M17, 32Q30, 53C15

1. Introduction

We will work in the complex analytic setting: all geometric objects refer to holomorphic ones.

Let V be a complex vector space and let M be a complex manifold with $\dim M = \dim V$. Let $\mathrm{Fr}(M)$ be the frame bundle whose fiber at $x \in M$ is $\mathrm{Fr}_x(M) = \mathrm{Isom}(V, T_x(M))$, the set of linear isomorphisms from V to the tangent space $T_x(M)$. This is a $\mathbf{GL}(V)$-principal bundle on M. Recall that for a connected algebraic subgroup $G \subset \mathbf{GL}(V)$, a *G-structure* on M means a G-principal subbundle of $\mathrm{Fr}(M)$, i.e., a reduction of the structure group of $\mathrm{Fr}(M)$ to G.

Recall that when M is a projective algebraic manifold, we say that M is uniruled if it is covered by rational curves, i.e., covered by images of generically injective holomorphic maps from $\mathbb{P}^1$ to M. In [4] (Main Theorem and Proposition 9), the following result was proved.

Theorem 1.1. *Let $G \subset \mathbf{GL}(V)$ be a connected algebraic subgroup that acts irreducibly on V. Suppose that there exists a G-structure on a uniruled projective manifold. Then*

(1) *either G acts transitively on $\mathbb{P}V$, or*

The author is supported by National Researcher Program 2010-0020413 of NRF.

(2) *the representation $G \subset \mathbf{GL}(V)$ is isomorphic to the isotropy representation of a compact irreducible Hermitian symmetric space of rank ≥ 2.*

It is worth mentioning that an obvious converse statement holds for (2): a compact irreducible Hermitian symmetric space is a uniruled projective manifold and it has a G-structure arising from the irreducible isotropy representation.

Recall (e.g. p. 545 of [3] or (1.1) in [5]) that for a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, its k-th prolongation

$$\mathfrak{g}^{(k)} \subset \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V)$$

consists of homomorphisms $A : \mathrm{Sym}^{k+1} V \to V$ such that for any $u_1, \ldots u_k \in V$, the endomorphism $A_{u_1 \cdots u_k} \in \mathfrak{gl}(V)$ defined by

$$A_{u_1 \cdots u_k}(v) = A(u_1, \ldots, u_k, v)$$

belongs to $\mathfrak{g}$. When $\mathfrak{g} \subset \mathfrak{gl}(V)$ is the Lie algebra of an irreducible representation $G \subset \mathbf{GL}(V)$, Kobayashi-Nagano [6] showed that $\mathfrak{g}^{(1)} \neq 0$ if and only if G is (1) or (2) of Theorem 1.1. Thus we can reformulate Theorem 1.1 as follows.

Theorem 1.2. *Let $G \subset \mathbf{GL}(V)$ be a connected algebraic subgroup that acts irreducibly on V. Suppose that there exists a G-structure on a uniruled projective manifold. Then the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ of G has a nonzero first prolongation.*

Our first result is a generalization of Theorem 1.2 to an arbitrary (not necessarily reductive) connected algebraic subgroup $G \subset \mathbf{GL}(V)$ and a more general complex manifold M in the following way.

Theorem 1.3. *Let $G \subset \mathbf{GL}(V)$ be a connected algebraic subgroup and let M be a complex manifold admitting an immersion $h : \mathbb{P}^1 \to M$. Suppose there exists a G-structure on M. Then the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ of G has a nonzero first prolongation.*

Theorem 1.3 covers Theorem 1.2 because any uniruled projective manifold M admits an immersion $\mathbb{P}^1 \to M$. The proof of Theorem 1.2 in [4] uses the highest weight orbit of the irreducible representation $G \subset \mathbf{GL}(V)$ and deformation theory of rational curves. So the reductiveness of G and the uniruledness of the base manifold were crucial for the arguments. Our proof of Theorem 1.3 is completely different and uses only standard tools of the theory of G-structures.

Unlike the result of Kobayashi-Nagano for the irreducible case, no reasonable classification is known for general Lie subalgebras of $\mathfrak{gl}(V)$ admitting nonzero prolongations. For Lie subalgebras of $\mathfrak{gl}(V)$ that

arise as automorphisms of nonsingular nondegenerate subvarieties of $\mathbb{P}V$, however, those admitting nonzero prolongations are classified in [1] and [2]. Combining these classification results with Theorem 1.3, we will prove the following. Note that Theorem 1.1 corresponds to the case (i) in Corollary 1.4.

Corollary 1.4. *Let $Z \subset \mathbb{P}V$ be a nonsingular nondegenerate subvariety of $\mathbb{P}V$. Let M be a complex manifold admitting an immersion $\mathbb{P}^1 \to M$. Assume that there exists a submanifold $\mathcal{C} \subset \mathbb{P}T(M)$ of the projectivized tangent bundle of M such that the fiber $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is projectively equivalent to $Z \subset \mathbb{P}V$ for every point $x \in M$. Then Z must be one of the followings.*

(i) *VMRT of an irreducible Hermitian symmetric space of rank ≥ 2 (explained in Example 4.4 of* [2]*)*
(ii) *VMRT of a (both even and odd) symplectic Grassmannian (explained in Example 4.5 of* [2]*)*
(iii) *a nonsingular linear section of* $\mathrm{Gr}(2, \mathbb{C}^5) \subset \mathbb{P}^9$ *of dimension 4 or 5*
(iv) *some nonsingular linear section of the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of dimension 7, 8 or 9*
(v) *certain biregular projections of varieties in (i) or (ii) (described in Section 4 of* [1]*).*

Our second result is a converse to Theorem 1.3, under some assumptions on $G \subset \mathbf{GL}(V)$.

Theorem 1.5. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be the Lie algebra of a connected algebraic group $G \subset \mathbf{GL}(V)$. Assume that*

(i) $\mathfrak{g}^{(1)} \neq 0$*;*
(ii) $\mathfrak{g}^{(k+1)} = 0$ *for some $k \geq 1$; and*
(iii) $\{v \in V,\ \mathfrak{g} \cdot v = 0\} = 0$.

Then there exists a homogenous quasi-projective variety equipped with a G-structure and covered by smooth rational curves.

Our construction of the homogeneous quasi-projective variety in Theorem 1.5 follows a standard argument in the theory of G-structures. The novelty lies in the existence of smooth rational curves on it.

As an application of Theorem 1.5, we will derive the following converse to Corollary 1.4.

Corollary 1.6. *Let $Z \subset \mathbb{P}V$ be one of the nonsingular nondegenerate varieties listed in (i)-(v) of Corollary 1.4. Then there exists a homogeneous quasi-projective variety M covered by smooth rational curves and equipped with a submanifold $\mathcal{C} \subset \mathbb{P}T(M)$ such that the fiber*

$\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is projectively equivalent to $Z \subset \mathbb{P}V$ for every point $x \in M$.

2. Proof of Theorem 1.3

We start recalling a few basic facts from the theory of G-structures.

Definition 2.1. Let V be a complex vector space and let $G \subset \mathbf{GL}(V)$ be a connected algebraic subgroup. Let M be a complex manifold with $\dim M = \dim V$ and let $\mathrm{Fr}(M)$ be the frame bundle whose fiber at $x \in M$ is $\mathrm{Fr}_x(M) = \mathrm{Isom}(V, T_x(M))$, the set of linear isomorphisms from V to the tangent space $T_x(M)$. Let $\mathcal{P} \subset \mathrm{Fr}(M)$ be a G-structure on M. Denote by $p : \mathcal{P} \to M$ the natural projection. For a point $\alpha \in \mathcal{P}$ and $x = p(\alpha)$, the composition of $p_* : T_\alpha(\mathcal{P}) \to T_x(M)$ and the isomorphism $\alpha^{-1} : T_x(M) \to V$, gives a natural homomorphism $\theta_\alpha : T_\alpha(\mathcal{P}) \to V$. This defines a natural V-valued 1-form θ on $\mathcal{P}$, called the *soldering form* on $\mathcal{P}$.

The following lemma is (2.5) in p. 310 of [7].

Lemma 2.2. *Given an element $A \in \mathfrak{g}$, let $\widetilde{A}$ be the fundamental vector field on $\mathcal{P}$ generated by A (via the right G-action on $\mathcal{P}$). The soldering form θ satisfies $\mathrm{d}\theta(\widetilde{A}, w) = -A \cdot \theta(w)$ for any $w \in T(\mathcal{P})$.*

Definition 2.3. In the setting of Definition 2.1, let $T^p \subset T(\mathcal{P})$ be the vertical subbundle given by the fibers of p. For a point $\alpha \in \mathcal{P}$ and a subspace $H \subset T_\alpha(\mathcal{P})$ complementary to T^p_α, we have an element $\Pi_H \in \mathrm{Hom}(\wedge^2 V, V)$ defined as follows. Writing $x = p(\alpha)$, we have two isomorphisms

$$V \xrightarrow{\alpha} T_x(M) \xleftarrow{p_*} H.$$

For $u, v \in V$, let $u^H, v^H \in H$ be the corresponding elements by these two isomorphisms. Then we define

$$\Pi_H(u, v) := \mathrm{d}\theta(u^H, v^H).$$

Definition 2.4. For a point $\alpha \in \mathcal{P}$, let H and H' be two subspaces of $T_\alpha(\mathcal{P})$ complementary to T^p_α. Then we have an element $s_{H,H'} \in \mathrm{Hom}(V, \mathfrak{g})$ defined in the following way. For $u \in V$, we have tangent vectors $u^H \in H, u^{H'} \in H'$ as defined in Definition 2.3. Then $u^H - u^{H'} \in T^p_\alpha$. We define $s_{H,H'}(u) \in \mathfrak{g}$ as the unique element such that the fundamental vector field on $\mathcal{P}$ generated by $s_{H,H'}(u)$ has value $u^H - u^{H'}$ at α.

The following is immediate from the definition.

Lemma 2.5. *In Definition 2.4, fix $H \subset T_\alpha(\mathcal{P})$ complementary to T^p_α. Then the map $H' \mapsto s_{H,H'} \in \mathrm{Hom}(V, \mathfrak{g})$ gives an identification of the set of complementary subspaces of $T^p_\alpha \subset T_\alpha(\mathcal{P})$ and $\mathrm{Hom}(V, \mathfrak{g})$.*

The following lemma is straight-forward from the definition of prolongations.

Lemma 2.6. *For a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, let $\delta : \mathrm{Hom}(V, \mathfrak{g}) \to \mathrm{Hom}(\wedge^2 V, V)$ be the homomorphism that sends $f \in \mathrm{Hom}(V, \mathfrak{g})$ to $\delta f \in \mathrm{Hom}(\wedge^2 V, V)$ defined by*

$$\delta f(u, v) := f(u) \cdot v - f(v) \cdot u.$$

Then $\mathfrak{g}^{(1)} = 0$ if and only if the homomorphism δ is injective.

The following is contained in Proposition 4.3 of [3]. Since our situation is simpler than the setting of [3], we give a full proof for the reader's convenience.

Lemma 2.7. *In Definition 2.3 and Definition 2.4, we have $\Pi_{H'} - \Pi_H = \delta s_{H,H'}$ where δ is as in Lemma 2.6.*

Proof. For $u, v \in V$, let $\widetilde{s_u}$ (resp. $\widetilde{s_v}$) be the fundamental vector field on $\mathcal{P}$ corresponding to $s_u := s_{H,H'}(u) \in \mathfrak{g}$ (resp. $s_v := s_{H,H'}(v) \in \mathfrak{g}$). Applying Lemma 2.2, we have

$$\begin{aligned}
\Pi_{H'}(u, v) - \Pi_H(u, v) &= \mathrm{d}\theta(u^{H'}, v^{H'}) - \mathrm{d}\theta(u^H, v^H) \\
&= \mathrm{d}\theta((u^{H'} - u^H), v^{H'}) + \mathrm{d}\theta(u^H, (v^{H'} - v^H)) \\
&= -\mathrm{d}\theta(\widetilde{s_u}, v^{H'}) - \mathrm{d}\theta(u^H, \widetilde{s_v}) \\
&= s_{H,H'}(u) \cdot v - s_{H,H'}(v) \cdot u \\
&= \delta s_{H,H'}(u, v).
\end{aligned}$$

This finishes the proof. ∎

Theorem 2.8. *Let $G \subset \mathbf{GL}(V)$ be a closed connected subgroup such that its Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ satisfies $\mathfrak{g}^{(1)} = 0$. Let $\mathcal{P} \subset \mathrm{Fr}(M)$ be a G-structure on a complex manifold M. Let $p : \mathcal{P} \to M$ be the natural projection and let $T^p \subset T(\mathcal{P})$ be the vertical subbundle given by the fibers of p. Then there exists a subbundle $\mathcal{H} \subset T(\mathcal{P})$ inducing a splitting $T(\mathcal{P}) \cong T^p \oplus \mathcal{H}$.*

Remark 2.9. The distribution $\mathcal{H}$ in Theorem 2.8 is *not* a connection on the principal bundle $\mathcal{P}$ (in the sense of Definition 1.2 in p. 298 of [7]), because it is not necessarily equivariant under the right G-action on $\mathcal{P}$.

Proof of Theorem 2.8. The assumption $\mathfrak{g}^{(1)} = 0$ implies that the homomorphism δ in Lemma 2.6 is injective. We fix a subspace $W \subset \mathrm{Hom}(\wedge^2 V, V)$ complementary to $\mathrm{Im}(\delta)$ once and for all. (This W may not be stable under the natural G-action on $\mathrm{Hom}(\wedge^2 V, V)$.)

Let $\alpha \in \mathcal{P}$ be a given point. We claim that there exists a unique subspace $\mathcal{H}_\alpha \subset T_\alpha(\mathcal{P})$ complementary to T^p_α such that $\Pi_{\mathcal{H}_\alpha} \in W$.

To see the existence, fix a complement H. Then there exists an element $s \in \mathrm{Hom}(V, \mathfrak{g})$ such that $\Pi_H + \delta s \in W$. By Lemma 2.5, we have $s = s_{H,H'}$ for some $H' \subset T_\alpha(\mathcal{P})$ complementary to T^p_α. Then Lemma 2.7 gives $\Pi_{H'} = \Pi_H + \delta s_{H,H'} \in W$. So we can put $\mathcal{H}_\alpha = H'$.

To see the uniqueness, if $\Pi_H, \Pi_{H'} \in W$, then $\Pi_H - \Pi_{H'} \in W$, but $\Pi_H - \Pi_{H'} \in \mathrm{Im}(\delta)$ by Lemma 2.7. Thus we have $\Pi_H = \Pi_{H'}$ and $\delta s_{H,H'} = 0$. By the injectivity of δ, we have $s_{H,H'} = 0$, which implies $H = H'$ by Lemma 2.5.

By the claim, at each $\alpha \in \mathcal{P}$, we have a unique horizontal subspace $\mathcal{H}_\alpha \subset T_\alpha(\mathcal{P})$. This defines a distribution $\mathcal{H} \subset T(\mathcal{P})$ complementary to T^p. □

Proof of Theorem 1.3. Let $G \subset \mathbf{GL}(V)$ be a closed connected subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$. Let M be a complex manifold admitting an immersion $h : \mathbb{P}^1 \to M$ and a G-structure $p : \mathcal{P} \to M$. Assuming $\mathfrak{g}^{(1)} = 0$, we derive a contradiction as follows.

Let $q : h^*\mathcal{P} \to \mathbb{P}^1$ be the G-principal bundle on $\mathbb{P}^1$ obtained by pulling back $p : \mathcal{P} \to M$ by h. By Theorem 2.8, we have a horizontal distribution $\mathcal{H} \subset T(\mathcal{P})$. Since h is an immersion, this distribution $\mathcal{H}$ induces a horizontal distribution $\mathcal{F} \subset T(h^*\mathcal{P})$, i.e., a rank-1 foliation complementary to the fibers of q. For a point $\alpha \in h^*\mathcal{P}$, the leaf $\mathcal{F}(\alpha)$ of the foliation $\mathcal{F}$ through α gives an unramified covering of $\mathbb{P}^1$. Thus $\mathcal{F}(\alpha)$ gives a holomorphic section $\sigma : \mathbb{P}^1 \to h^*\mathcal{P} \subset h^*\,\mathrm{Fr}(M)$ of the principal bundle $h^*\,\mathrm{Fr}(M) \to \mathbb{P}^1$. Then the family of isomorphisms

$$\{\sigma(x) \in \mathrm{Fr}_{h(x)}(M) = \mathrm{Isom}(V, T_{h(x)}(M)), x \in \mathbb{P}^1\}$$

shows that the pull-back bundle $h^*T(M)$ is a trivial vector bundle on $\mathbb{P}^1$.

We have an injection $\mathrm{d}h : T(\mathbb{P}^1) \subset h^*T(M)$ by the immersion h. Since $T(\mathbb{P}^1)$ has a nonzero holomorphic section vanishing at a point of $\mathbb{P}^1$, the vector bundle $h^*T(M)$ has a nonzero holomorphic section vanishing at a point. This contradicts the triviality of $h^*T(M)$. □

3. Proof of Theorem 1.5

Given a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ with $\mathfrak{g}^{(k+1)} = 0$ and $\mathfrak{g}^{(k)} \neq 0$ for some $k \geq 1$, the direct sum

$$\widetilde{\mathfrak{g}} := V + \mathfrak{g} + \mathfrak{g}^{(1)} + \cdots \mathfrak{g}^{(k)}$$

has a natural structure of a graded Lie algebra, with the weight -1 for V, the weight 0 for $\mathfrak{g}$ and the weight j for $\mathfrak{g}^{(j)}$ for each $1 \leq j \leq k$, as described in p. 545 of [3]. Write

$$\begin{aligned} \mathfrak{p} &= \mathfrak{g} + \mathfrak{g}^{(1)} + \cdots + \mathfrak{g}^{(k)} \\ \mathfrak{p}_+ &= \mathfrak{g}^{(1)} + \cdots + \mathfrak{g}^{(k)} \end{aligned}$$

For convenience, we will also write W for the underlying vector space of the Lie algebra $\widetilde{\mathfrak{g}}$. By the condition (iii) of Theorem 1.5, the adjoint representation of $\widetilde{\mathfrak{g}}$ is faithful. Thus we may regard $\widetilde{\mathfrak{g}}$ as a subalgebra of $\mathfrak{gl}(W)$. Since the subalgebras V and $\mathfrak{p}_+$ of $\widetilde{\mathfrak{g}}$ are nilpotent, they are Lie algebras of algebraic subgroups of $\mathrm{GL}(W)$. Combining this with the fact that $G \subset \mathbf{GL}(V)$ is an algebraic subgroup, we see (e.g. by Corollary 24.5.10 in [8]) that the subgroups

$$P \subset \widetilde{G} \subset \mathrm{GL}(W)$$

corresponding to the subalgebras

$$\mathfrak{p} \subset \widetilde{\mathfrak{g}} \subset \mathfrak{gl}(W)$$

are algebraic subgroups. Thus the coset $M = \widetilde{G}/P$ is a homogeneous quasi-projective variety.

We claim that M is equipped with a G-structure $\mathcal{P} \subset \mathrm{Fr}(M)$ defined as follows. At the base point $o \in M$ corresponding to P, we have a natural identification $T_o(M) = V$. By the action of $\widetilde{G}$ on M, an element $g \in \widetilde{G}$ induces an isomorphism $g_* : V \to T_x(M)$ of V and the tangent space at $x = [gP] \in M$. Then

$$\mathrm{Fr}_x(M) = \mathrm{Isom}(V, T_x(M))$$

contains the submanifold $\mathcal{P}_x$ consisting of $\alpha \in \mathrm{Isom}(V, T_x(M))$ such that

$$g_*^{-1} \circ \alpha \in \mathrm{Isom}(V, V) = \mathbf{GL}(V)$$

belongs to $G \subset \mathbf{GL}(V)$. Note that if $g \in P$, then $g_* : V \to T_o(M) = V$ belongs to G. Thus the submanifold $\mathcal{P}_x$ depends only on $x \in M$, not on the choice of $g \in G$ satisfying $x = [gP]$. This defines the G-structure $\mathcal{P}$ on M.

It remains to show that M contains a nonsingular rational curve. Setting $m = \dim \mathfrak{p}$, consider the action of $\widetilde{G}$ on $\wedge^m W$ induced by the

adjoint action on W. This gives an action of $\widetilde{G}$ on the Grassmannian $\mathrm{Gr}(m, W) \subset \mathbb{P} \wedge^m W$. Then the point

$$\mathbb{P}\det(\mathfrak{p}) = \mathbb{P} \wedge^m \mathfrak{p} \in \mathrm{Gr}(m, W)$$

is fixed by P. From the Lie algebra structure of $\widetilde{\mathfrak{g}}$ (described in p. 545 of [3]), we see that $P \subset \widetilde{G}$ is exactly the isotropy subgroup of this point. Thus M is embedded in $\mathrm{Gr}(m, W)$ as the orbit of $\widetilde{G}$ through the point $\mathbb{P}\det(\mathfrak{p})$.

Let us first consider the case $k = 1$. We can pick a nonzero element $A \in \mathfrak{g}^{(1)}$ and an element $v \in V$ such that $u := [A, v] \in \mathfrak{g}$ is not zero. Choose a basis $\{p_1, \ldots, p_m\}$ of $\mathfrak{p}$ such that

(1) $p_m = u$;
(2) $\{p_1, \ldots, p_\ell\}$ is a basis of $\mathfrak{p}_+$ for some $\ell < m$; and
(3) $\{p_{\ell+1}, p_{\ell+2}, \ldots, p_{m-1}, p_m\}$ is a basis of $\mathfrak{g}$.

Let $\{g_t := \exp(tA) \in \widetilde{G}, t \in \mathbb{C}\}$ be the additive subgroup of $\widetilde{G}$ generated by A. From the fact that $\widetilde{\mathfrak{g}}$ is a graded Lie algebra, we have

$$\begin{aligned} 0 &= [A, p_i] \text{ for } 1 \leq i \leq \ell \\ q_j &:= [A, p_j] \ \in \mathfrak{p}_+ \text{ for } \ell + 1 \leq j \leq m \\ 0 &= [A, [A, q_j]] \text{ for } \ell + 1 \leq j \leq m \end{aligned}$$

Then for all $t \in \mathbb{C}$,

$$\begin{aligned} g_t \cdot p_i &= p_i \text{ for } 1 \leq i \leq \ell \\ g_t \cdot p_j &= p_j + tq_j \text{ for } \ell + 1 \leq j \leq m - 1 \\ g_t \cdot v &= v + tp_m + \frac{t^2}{2} q_m. \end{aligned}$$

Thus $g_t \cdot (p_1 \wedge \cdots \wedge p_{m-1} \wedge v)$ becomes

$$p_1 \wedge \cdots \wedge p_\ell \wedge (p_{\ell+1} + tq_{\ell+1}) \wedge \cdots \wedge (p_{m-1} + tq_{m-1}) \wedge (v + tp_m + t^2 q_m).$$

Since $p_1, \ldots, p_\ell$ form a basis of $\mathfrak{p}_+ = \mathfrak{g}^{(1)}$ and $q_j \in \mathfrak{p}_+$, this reduces to

$$\begin{aligned} &p_1 \wedge \cdots \wedge p_\ell \wedge p_{\ell+1} \wedge \cdots \wedge p_{m-1} \wedge (v + tp_m) \\ &= (p_1 \wedge \cdots \wedge p_{m-1} \wedge v) + t(p_1 \wedge \cdots \wedge p_m). \end{aligned}$$

It follows that the closure of the orbit

$$\{[g_t \cdot (p_1 \wedge \cdots \wedge p_{m-1} \wedge v)] \in \mathrm{Gr}(m, W), t \in \mathbb{C}\}$$

is a line in $\mathbb{P} \wedge^m W$ lying on M, containing the point $\mathbb{P}\det(\mathfrak{p})$. Thus Theorem 1.5 is proved when $k = 1$.

Now consider the case $k \geq 2$. We can modify the above argument for $k = 1$ in the following way. From the assumption $\mathfrak{g}^{(k)} \neq 0$, we can pick a nonzero element $A \in \mathfrak{g}^{(k)}$ and an element $v \in V$ such that

$u := [A, v] \in \mathfrak{g}^{(k-1)}$ is not zero. Choose a basis $\{p_1, \dots, p_m\}$ of $\mathfrak{p}$ such that

(1) $p_m = u \ \in \mathfrak{p}_+$;
(2) $\{p_1, \dots, p_\ell, p_m\}$ is a basis of $\mathfrak{p}_+$ for some $\ell < m$;
(3) $\{p_{\ell+1}, p_{\ell+2}, \dots, p_{m-1}\}$ is a basis of $\mathfrak{g}$; and
(4) $\{p_1, \dots, p_n\}$ is a basis of $\mathfrak{g}^{(k)}$ for some $n \leq \ell$.

Let $\{g_t := \exp(tA) \in \widetilde{G}, t \in \mathbb{C}\}$ be the additive subgroup of $\widetilde{G}$ generated by A. From the fact that $\widetilde{\mathfrak{g}}$ is a graded Lie algebra, we have elements $q_j = [A, p_j]$ for $\ell+1 \leq j \leq m-1$, belonging to $\mathfrak{g}^{(k)}$, such that for all $t \in \mathbb{C}$,

$$\begin{aligned} g_t \cdot p_i &= p_i \text{ for } 1 \leq i \leq \ell \\ g_t \cdot p_j &= p_j + tq_j \text{ for } \ell+1 \leq j \leq m-1 \\ g_t \cdot v &= v + tp_m. \end{aligned}$$

Then $g_t \cdot (p_1 \wedge \cdots \wedge p_{m-1} \wedge v)$ becomes

$$(p_1 \wedge \cdots \wedge p_\ell) \wedge (p_{\ell+1} + tq_{\ell+1}) \wedge \cdots \wedge (p_{m-1} + tq_{m-1}) \wedge (v + tp_m).$$

Since $p_1, \dots, p_n$ form a basis of $\mathfrak{g}^{(k)}$ and $q_j \in \mathfrak{g}^{(k)}$, this reduces to

$$\begin{aligned} & p_1 \wedge \cdots \wedge p_\ell \wedge p_{\ell+1} \wedge \cdots \wedge p_{m-1} \wedge (v + tp_m) \\ &= (p_1 \wedge \cdots \wedge p_{m-1} \wedge v) + t(p_1 \wedge \cdots \wedge p_m). \end{aligned}$$

It follows that the closure of the orbit

$$\{[g_t \cdot (p_1 \wedge \cdots \wedge p_{m-1} \wedge v)] \in \mathrm{Gr}(m, W), t \in \mathbb{C}\}$$

is a line in $\mathbb{P} \wedge^m W$ lying on M, containing the point $\mathbb{P}\det(\mathfrak{p})$. This completes the proof of Theorem 1.5.

4. Proof of Corollaries 1.4 and 1.6

Given a nonsingular nondegenerate projective variety $Z \subset \mathbb{P}V$, let $\widehat{Z} \subset V$ be its affine cone. Let $G \subset \mathrm{GL}(V)$ be the connected component of the linear automorphism group of $\widehat{Z}$ and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be its Lie algebra.

Proof of Corollary 1.4. Let $G^+ \subset \mathrm{GL}(V)$ be the linear automorphism group of $\widehat{Z}$, so that G is a connected component of G^+. Given $\mathcal{C} \subset \mathbb{P}T(M)$ as in Corollary 1.4, we define a G^+-structure $\mathcal{P}^+ \subset \mathrm{Fr}(M)$ as follows. At a point $x \in M$, the fiber $\mathcal{P}^+_x \subset \mathrm{Fr}_x(M)$ is the set of isomorphisms from V to $T_x(M)$ that sends $Z \subset \mathbb{P}V$ to $\mathcal{C}_x \subset \mathbb{P}T_x(M)$. This is certainly a G^+-structure.

Now given an immersion $h : \mathbb{P}^1 \to M$, we can find a germ of complex manifold $\mathbb{P}^1 \subset M^o, \dim M^o = \dim M$ with an immersion $M^o \to M$, by taking the union of germs of points on $h(\mathbb{P}^1)$ in M. This complex

manifold M^o is equipped with the G^+-structure induced by $\mathcal{P}^+$. Since $\mathbb{P}^1$ is simply connected, a component of $\mathcal{P}^+$ gives a G-structure on M^o. Thus Theorem 1.3 implies that $\mathfrak{g}^{(1)} \neq 0$. Finally, Theorem 7.13 of [2] says that those listed in (i)-(v) of Corollary 1.4 are the only nonsingular nondegenerate projective varieties with $\mathfrak{g}^{(1)} \neq 0$. □

Proof of Corollary 1.6. For the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ associated with a nonsingular nondegenerate $Z \subset \mathbb{P}V$, the vanishing of $\mathfrak{g}^{(2)}$ is proved in Theorem 1.1.2 of [5] and the condition (iii) of Theorem 1.5 is immediate because $\mathfrak{g}$ contains $\mathbb{C}\cdot\mathrm{Id}_V$. Thus we can construct M with a G-structure as in Theorem 1.5, which we denote by $\mathcal{P} \subset \mathrm{Fr}(M)$. Define $\mathcal{C} \subset \mathbb{P}T(M)$ as follows. For $x \in M$, let $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ be the image of $Z \subset \mathbb{P}V$ under an element of $\alpha \in \mathcal{P}_x \subset \mathrm{Isom}(V, T_x(M))$. If $\beta \in \mathcal{P}_x$, then $\beta = \alpha \circ g$ for some element $g \in G$. Since g sends Z to itself, the β-image of Z agrees with the α-image of Z. Thus $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is well-defined, independent of α, depending only on x. □

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